

Affine Schottky Groups and Crooked Tilings

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In his doctoral thesis [3] and subsequent papers [4, 5], Todd Drumm developed a theory of fundamental domains for discrete groups of isometries of Minkowski $(2 + 1)$ -space \mathbb{E} , using polyhedra called *crooked planes* and *crooked half-spaces*. This paper expounds these results.

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Main Theorem. *Let $\mathcal{C}_1^-, \mathcal{C}_1^+, \dots, \mathcal{C}_m^-, \mathcal{C}_m^+ \subset \mathbb{E}$ be a family of crooked planes bounding crooked half-spaces $\mathcal{H}_1^-, \mathcal{H}_1^+, \dots, \mathcal{H}_m^-, \mathcal{H}_m^+ \subset \mathbb{E}$ and $h_1, \dots, h_m \in \text{Isom}^0(\mathbb{E})$ such that:*

- *any pair of the \mathcal{H}_i^j are pairwise disjoint;*
- *$h_i(\mathcal{H}_i^-) = \mathbb{E} - \mathcal{H}_i^+$.*

Then:

- *the group Γ generated by $h_1, \dots, h_m \in \text{Isom}^0(\mathbb{E})$ acts properly discontinuously on \mathbb{E} ;*
- *the polyhedron*

$$X = \mathbb{E} - \bigcup_{i=1}^m (\bar{\mathcal{H}}_i^+ \cup \bar{\mathcal{H}}_i^-)$$

is a fundamental domain for the Γ -action on \mathbb{E} .

The first examples of properly discontinuous affine actions of nonabelian free groups were constructed by Margulis [14, 15], following a suggestion of Milnor [16]. (Compact quotients of affine space were classified by Fried and Goldman [12] about the same time.) For background on this problem, we refer the reader to these articles as well as the survey articles [2], [6].

Here is the outline of this paper. In §1, we collect basic facts about the geometry of Minkowski space and $\mathbb{R}^{2,1}$. In §1.5, we prove a basic lemma on how isometries compress Euclidean balls in special directions. A key idea is the *hyperbolicity* of a hyperbolic isometry, motivated by ideas of Margulis [14, 15], and discussed in §1.4, using the *null frames* associated with spacelike vectors and hyperbolic isometries. The hyperbolicity of a hyperbolic element g is defined as the distance between the attracting and repelling directions, and g is ϵ -hyperbolic if its hyperbolicity is $\geq \epsilon$.

§2 reviews Schottky subgroups of $\text{SO}^0(2, 1)$ acting on $\mathbb{H}_{\mathbb{R}}^2$. This both serves as the prototype for the subsequent discussion of affine Schottky groups and as the starting point for the construction of affine Schottky groups. For Schottky groups acting on $\mathbb{H}_{\mathbb{R}}^2$, a completeness argument in Poincaré's polygon theorem shows that the images of the fundamental polygon tile all of $\mathbb{H}_{\mathbb{R}}^2$; the analogous result for affine Schottky groups is the main topic of this paper.

§2.3 gives a criterion for ϵ -hyperbolicity of elements of Schottky groups.

§3 introduces *crooked planes* as the analogs of geodesics in $\mathbb{H}_{\mathbb{R}}^2$ bounding the fundamental polygon. §3.1 extends Schottky groups to linear actions on $\mathbb{R}^{2,1}$. In §3.4, affine deformations of these linear actions are proved to be properly discontinuous on open subsets of Minkowski space, using the standard argument for Schottky groups. The fundamental polyhedron X is bounded by crooked planes, in exactly the same configuration as the geodesics bounding the fundamental polygon for Schottky groups. The generators of the affine Schottky group pair the faces of X in exactly the same pattern as for the original Schottky group.

The difficult part of the proof is to show that the images $\gamma\bar{X}$ tile *all* of Minkowski space. Assuming that a point p lies in $\mathbb{E} - \Gamma\bar{X}$, Drumm intersects the tiling with a fixed definite plane $P \ni p$. In §3.3, we abstract this idea by introducing *zigzags* and *zigzag regions*, which are the intersections of crooked planes and half-spaces with P .

The proof that $\Gamma\bar{X} = \mathbb{E}$ (that is, *completeness* of the affine structure on the quotient $(\Gamma\bar{X})/\Gamma$) involves a nested sequence of crooked half-spaces \mathfrak{H}_k containing p . This sequence is constructed in §4.1 and is indexed by a sequence $\gamma_k \in \Gamma$. §4.2

gives a uniform lower bound to the Euclidean width of \bar{X} . Bounding the uniform width is a key ingredient in proving completeness for Schottky groups in hyperbolic space.

§4.3 approximates the nested sequence of zigzag regions by a nested sequence of half-planes Π_k in P . The uniform width bounds are used to prove Lemma 4.3, on the existence of tubular neighborhoods T_k of $\partial\Pi_k$ which are disjoint. The Compression Lemma 1.8, combined with the observation (Lemma 4.4) that the zigzag regions do not point too far away from the direction of expansion of the γ_k , imply that the Euclidean width of the T_k are bounded below in terms of the hyperbolicity of γ_k . (Lemma 4.5). Thus it suffices to find $\epsilon > 0$ such that infinitely many γ_k are ϵ -hyperbolic.

§4.5 applies the criterion for ϵ -hyperbolicity derived in §2.3 (Lemma 2.7) to find $\epsilon_0 > 0$ guaranteeing ϵ_0 -hyperbolicity in many cases. In the other cases, the sequence γ_k has a special form, the analysis of which gives a smaller ϵ such that every γ_k is now ϵ -hyperbolic. The details of this constitute §4.6.

We follow Drumm's proof, with several modifications. We wish to thank Todd Drumm for the inspiration for this work and Maria Morrill, as well as Todd Drumm, for several helpful conversations.

1. Minkowski space

We begin with background on $\mathbb{R}^{2,1}$ and Minkowski (2+1)-space \mathbb{E} . $\mathbb{R}^{2,1}$ is defined as a real inner product space of dimension 3 with a nondegenerate inner product of index 1. Minkowski space is an affine space \mathbb{E} whose underlying vector space is $\mathbb{R}^{2,1}$; equivalently \mathbb{E} is a simply-connected geodesically complete flat Lorentzian manifold. If $p, q \in \mathbb{E}$, then a unique vector $\mathbf{v} \in \mathbb{R}^{2,1}$ represents the *displacement* from p to q , that is, translation by the vector \mathbf{v} is the unique translation taking p to q ; we write

$$\mathbf{v} = q - p \text{ and } q = p + \mathbf{v}.$$

Lines and planes in $\mathbb{R}^{2,1}$ are classified in terms of the inner product. The identity component $\text{SO}^0(2,1)$ of the group of linear isometries of $\mathbb{R}^{2,1}$ comprises linear transformations preserving the *future* component \mathcal{N}_+ of the set \mathcal{N} of timelike vectors, as well as orientation. The set of rays in \mathcal{N}_+ is a model for *hyperbolic 2-space* $\mathbb{H}_{\mathbb{R}}^2$, and the geometry of $\mathbb{H}_{\mathbb{R}}^2$ serves as a model for the geometry of $\mathbb{R}^{2,1}$ and \mathbb{E} .

A key role is played by the *ideal boundary* of $\mathbb{H}_{\mathbb{R}}^2$. Its intrinsic description is as the projectivization of the lightcone in $\mathbb{R}^{2,1}$, although following Margulis [14, 15] we identify it with a section on a Euclidean sphere. However, to simplify the formulas, we find it more convenient to take for this section the intersection of the future-pointing lightcone with the sphere $S^2(\sqrt{2})$ of radius $\sqrt{2}$, rather than the Euclidean unit sphere, which is used in the earlier literature. We hope this departure from tradition justifies itself by the resulting simplification of notation.

1.1. Minkowski space and its projectivization. Let $\mathbb{R}^{2,1}$ be the three-dimensional real vector space \mathbb{R}^3 with the inner product

$$\mathbb{B}(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 - u_3v_3$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^{2,1}.$$

Definition 1.1. A vector \mathbf{u} is said to be

- spacelike if $\mathbb{B}(\mathbf{u}, \mathbf{u}) > 0$,
- lightlike (or null) if $\mathbb{B}(\mathbf{u}, \mathbf{u}) = 0$,
- timelike if $\mathbb{B}(\mathbf{u}, \mathbf{u}) < 0$.

The union of all null lines is called the lightcone.

Here is the dual trichotomy for planes in $\mathbb{R}^{2,1}$:

Definition 1.2. A 2-plane $P \subset \mathbb{R}^{2,1}$ is:

- indefinite if the restriction of \mathbb{B} to P is indefinite, or equivalently if $P \cap \mathcal{N} \neq \emptyset$;
- null if $P \cap \bar{\mathcal{N}}$ is a null line;
- definite if the restriction $\mathbb{B}|_P$ is positive definite, or equivalently if $P \cap \bar{\mathcal{N}} = \{0\}$.

Let $O(2, 1)$ denote the group of linear isometries of $\mathbb{R}^{2,1}$ and $SO(2, 1) = O(2, 1) \cap SL(3, \mathbb{R})$ as usual. Let $SO^0(2, 1)$ denote the identity component of $O(2, 1)$ (or $SO(2, 1)$). The affine space with underlying vector space $\mathbb{R}^{2,1}$ has a complete flat Lorentzian metric arising from the inner product on $\mathbb{R}^{2,1}$; we call it *Minkowski space* and denote it by \mathbb{E} . Its isometry group $\text{Isom}(\mathbb{E})$ of \mathbb{E} splits as a semidirect product $O(2, 1) \ltimes V$ (where V denotes the vector group of translations of \mathbb{E}) and its identity component $\text{Isom}^0(\mathbb{E})$ of $\text{Isom}(\mathbb{E})$ is $SO^0(2, 1) \ltimes V$. The elements of $\text{Isom}(\mathbb{E})$ are called *affine isometries*, to distinguish them from the linear isometries in $O(2, 1)$.

The linear isometries can be characterized as the affine isometries fixing a chosen “origin”, which is used to identify the *points* of \mathbb{E} with the *vectors* in $\mathbb{R}^{2,1}$. The *origin* is the point which corresponds to the zero vector $\mathbf{0} \in \mathbb{R}^{2,1}$. Let $\mathbb{L} : \text{Isom}^0(\mathbb{E}) \rightarrow SO^0(2, 1)$ denote the homomorphism associating to an affine isometry its linear part.

Let $\mathbb{P}(\mathbb{R}^{2,1})$ denote the *projective space* associated to $\mathbb{R}^{2,1}$, that is the space of 1-dimensional linear subspaces of $\mathbb{R}^{2,1}$. Let

$$\mathbb{P} : \mathbb{R}^{2,1} - \{0\} \rightarrow \mathbb{P}(\mathbb{R}^{2,1})$$

denote the quotient mapping which associates to a nonzero vector the line it spans.

Let \mathcal{N} denote the set of all timelike vectors. Its two connected components are the convex cones

$$\mathcal{N}_+ = \{\mathbf{v} \in \mathcal{N} \mid v_3 > 0\}$$

and

$$\mathcal{N}_- = \{\mathbf{v} \in \mathcal{N} \mid v_3 < 0\}.$$

We call \mathcal{N}_+ the *future* or *positive time-orientation* of $\mathbf{0}$.

Hyperbolic 2-space $H_{\mathbb{R}}^2 = \mathbb{P}(\mathcal{N})$ consists of timelike lines in $\mathbb{R}^{2,1}$. The distance $d(u, v)$ between two points $u = \mathbb{P}(\mathbf{u}), v = \mathbb{P}(\mathbf{v})$ represented by timelike vectors $\mathbf{u}, \mathbf{v} \in \mathcal{N}$ is given by:

$$\cosh(d(u, v)) = \frac{|\mathbb{B}(\mathbf{u}, \mathbf{v})|}{\sqrt{\mathbb{B}(\mathbf{u}, \mathbf{u})\mathbb{B}(\mathbf{v}, \mathbf{v})}}.$$

The hyperbolic plane can be identified with either one of the components of the two-sheeted hyperboloid defined by $\mathbb{B}(\mathbf{v}, \mathbf{v}) = -1$. This hyperboloid is a section of $\mathbb{P} : \mathcal{N} \rightarrow \mathbb{H}_{\mathbb{R}}^2$, and inherits a complete Riemannian metric of constant curvature -1 from $\mathbb{R}^{2,1}$.

The group $\mathrm{SO}^0(2, 1)$ acts linearly on \mathcal{N} , and thus by projective transformations on $\mathbb{H}_{\mathbb{R}}^2$. It preserves the subsets \mathcal{N}_{\pm} and acts isometrically with respect to the induced Riemannian structures.

The boundary $\partial\mathcal{N}$ is the union of all null lines, that is the lightcone. The projectivization of $\partial\mathcal{N} - \{0\}$ is the *ideal boundary* $\partial\mathbb{H}_{\mathbb{R}}^2$ of hyperbolic 2-space.

Let \mathfrak{S} denote the set of unit-spacelike vectors :

$$\mathfrak{S} = \{\mathbf{v} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathbf{v}, \mathbf{v}) = 1\}.$$

It is a one-sheeted hyperboloid which is homeomorphic to an annulus. Points in \mathfrak{S} correspond to oriented geodesics in $\mathbb{H}_{\mathbb{R}}^2$, or geodesic half-planes in $\mathbb{H}_{\mathbb{R}}^2$ as follows. Let $\mathbf{v} \in \mathfrak{S}$. Then define the *half-plane* $H_{\mathbf{v}} = \mathbb{P}(\tilde{H}_{\mathbf{v}})$ where

$$\tilde{H}_{\mathbf{v}} = \{\mathbf{u} \in \mathcal{N}_+ \mid \mathbb{B}(\mathbf{u}, \mathbf{v}) > 0\}.$$

$H_{\mathbf{v}}$ is one of the two components of the complement $\mathbb{H}_{\mathbb{R}}^2 - l_{\mathbf{v}}$ where

$$(1.1) \quad l_{\mathbf{v}} = \mathbb{P}(\{\mathbf{u} \in \mathcal{N}_+ \mid \mathbb{B}(\mathbf{u}, \mathbf{v}) = 0\}) = \partial H_{\mathbf{v}}$$

is the geodesic in $\mathbb{H}_{\mathbb{R}}^2$ corresponding to the line $\mathbb{R}\mathbf{v}$ spanned by \mathbf{v} .

1.2. A little Euclidean geometry. Denote the Euclidean norm by

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v}_1)^2 + (\mathbf{v}_2)^2 + (\mathbf{v}_3)^2}$$

and let ρ denote the Euclidean distance on \mathbb{E} defined by

$$\rho(p, q) = \|p - q\|.$$

If $S \subset \mathbb{E}$ and $\delta > 0$, the *Euclidean δ -neighborhood* of S is $B(S, \delta) = \{y \in \mathbb{E} \mid \rho(S, y) < \delta\}$. Note that $B(S, \delta) = \bigcup_{x \in S} B(x, \delta)$.

Let

$$S^2(\sqrt{2}) = \{\mathbf{v} \in \mathbb{R}^{2,1} \mid \|\mathbf{v}\| = \sqrt{2}\}$$

be the Euclidean sphere of radius $\sqrt{2}$. Let S^1 denote the intersection $S^2(\sqrt{2}) \cap \partial\mathcal{N}_+$, consisting of points

$$\mathbf{u}_{\phi} = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \\ 1 \end{bmatrix}$$

where $\phi \in \mathbb{R}$. The subgroup of $\mathrm{SO}^0(2, 1)$ preserving S^1 and $S^2(\sqrt{2})$ is the subgroup $\mathrm{SO}(2)$ consisting of rotations

$$R_{\phi} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

While the linear action of $\mathrm{SO}^0(2, 1)$ does not preserve S^1 , we may use the identification of S^1 with $\mathbb{P}(\partial\mathcal{N}_+)$ to define an action of $\mathrm{SO}^0(2, 1)$ on S^1 . If $g \in \mathrm{SO}^0(2, 1)$, we denote this action by $\mathbb{P}(g)$, that is, if $\mathbf{u} \in S^1$, then $\mathbb{P}(g)(\mathbf{u})$ is the image of $\mathbb{P}(g(\mathbf{u}))$ under the identification $\mathbb{P}(\partial\mathcal{N}_+) \rightarrow S^1$. Throughout this paper (and especially in §2), we shall consider this action of $\mathrm{SO}^0(2, 1)$ on S^1 .

The restriction of either the Euclidean metric or the Lorentzian metric to S^1 is the rotationally invariant metric $d\phi^2$ on S^1 for which the total circumference is 2π .

Definition 1.3. An interval on S^1 is an open subset A of the form $\{u_\phi \mid \phi_1 < \phi < \phi_2\}$ where $\phi_1 < \phi_2$ and $\phi_2 - \phi_1 < 2\pi$. Its length $\Phi(A)$ is $\phi_2 - \phi_1$.

Note that if $\phi_1 < \phi_2$, the points $\mathbf{a}_1 = u_{\phi_1}$ and $\mathbf{a}_2 = u_{\phi_2} \in S^1$ bound two different intervals : we can either take $A = \{u_\phi \mid \phi_1 < \phi < \phi_2\}$ or $A = \{u_\phi \mid \phi_2 < \phi < \phi_1 + 2\pi\}$. The length of one of these intervals is less than or equal to π , in which case

$$\rho(\mathbf{a}_1, \mathbf{a}_2) = 2 \sin(\Phi(A)/2).$$

Intervals correspond to unit-spacelike vectors as follows. Let $A \subset S^1$ be an interval bounded by $\mathbf{a}_1 = u_{\phi_1}$ and $\mathbf{a}_2 = u_{\phi_2}$, where $\phi_1 < \phi_2$. Then the Lorentzian cross-product $u_2 \boxtimes u_1$ (see, for example, Drumm-Goldman [7, 8]) is a positive scalar multiple of the corresponding unit-spacelike vector.

In [3, 4, 5], Drumm considers conical neighborhoods in $\partial\mathcal{N}_+$ rather than intervals in S^1 . A *conical neighborhood* is a connected open subset U of the future lightcone $\partial\mathcal{N}_+$ which is invariant under the group \mathbb{R}_+ of positive scalar multiplications. The projectivization $\mathbb{P}(U)$ of a conical neighborhood is a connected open interval in $\mathbb{P}(\partial\mathcal{N}_+) \approx S^1$ which we may identify with the interval $U \cap S^1$, which is an interval. Thus every conical neighborhood equals $\mathbb{R}_+ \cdot A$, where $A \subset S^1$ is the interval $A = U \cap S^1$.

1.3. Null frames. Let $\mathbf{v} \in \mathfrak{S}$ be a unit-spacelike vector. We associate to \mathbf{v} two null vectors $\mathbf{x}^\pm(\mathbf{v})$ in the future which are \mathbb{B} -orthogonal to \mathbf{v} and lie on the unit circle $S^1 = S^2(\sqrt{2}) \cap \partial\mathcal{N}_+$. These vectors correspond to the endpoints of the geodesic l_v . To define $\mathbf{x}^\pm(\mathbf{v})$, first observe that the orthogonal complement

$$\mathbf{v}^\perp = \{u \in \mathbb{R}^{2,1} \mid \mathbb{B}(u, \mathbf{v}) = 0\}$$

meets the positive lightcone $\partial\mathcal{N}_+$ in two rays. Then $S^1 = \partial\mathcal{N}_+ \cap S^2(\sqrt{2})$ meets \mathbf{v}^\perp in a pair of vectors $\mathbf{x}^\pm(\mathbf{v})$. We determine which one of this pair is $\mathbf{x}^+(\mathbf{v})$ and which one is $\mathbf{x}^-(\mathbf{v})$ by requiring that the triple

$$(\mathbf{x}^-(\mathbf{v}), \mathbf{x}^+(\mathbf{v}), \mathbf{v})$$

be a positively oriented basis of $\mathbb{R}^{2,1}$. We call such a basis a *null frame* of $\mathbb{R}^{2,1}$. (Compare Figure 1.) The pair $\{\mathbf{x}^-(\mathbf{v}), \mathbf{x}^+(\mathbf{v})\}$ is a basis for the indefinite plane \mathbf{v}^\perp . In fact, \mathbf{v} is the unit-spacelike vector corresponding to the interval bounded by the ordered pair $(\mathbf{x}^+(\mathbf{v}), \mathbf{x}^-(\mathbf{v}))$.

Hyperbolic elements of $\mathrm{SO}^0(2, 1)$ also determine null frames. Recall that $g \in \mathrm{SO}^0(2, 1)$ is *hyperbolic* if it has real distinct eigenvalues, which are necessarily positive. Then the eigenvalues are $\lambda, 1, \lambda^{-1}$ and we may assume that

$$\lambda < 1 < \lambda^{-1}.$$

Let $\mathbf{x}^-(g)$ denote the unique eigenvector with eigenvalue λ lying on S^1 and $\mathbf{x}^+(g)$ denote the unique eigenvector with eigenvalue λ^{-1} lying on S^1 . Then $\mathbf{x}^0(g) \in \mathfrak{S}$ is the uniquely determined eigenvector of g such that $(\mathbf{x}^-(g), \mathbf{x}^+(g), \mathbf{x}^0(g))$ is positively oriented. Observe that this is a null frame, since $\mathbf{x}^\pm(g) = \mathbf{x}^\pm(\mathbf{v})$, where $\mathbf{v} = \mathbf{x}^0(g)$.

1.4. ϵ -Hyperbolicity. We may define the *hyperbolicity* of a unit-spacelike vector \mathbf{v} as the Euclidean distance $\rho(\mathbf{x}^+(\mathbf{v}), \mathbf{x}^-(\mathbf{v}))$. The following definition (Drumm-Goldman [7]) is based on Margulis [14, 15].

Definition 1.4. A unit-spacelike vector \mathbf{v} is ϵ -spacelike if $\rho(\mathbf{x}^+(\mathbf{v}), \mathbf{x}^-(\mathbf{v})) \geq \epsilon$. A hyperbolic element $g \in \mathrm{SO}^0(2, 1)$ is ϵ -hyperbolic if $\mathbf{x}^0(g)$ is ϵ -spacelike. An affine isometry is ϵ -hyperbolic if its linear part is an ϵ -hyperbolic linear isometry.

The spacelike vector \mathbf{v} corresponds to a geodesic $l_{\mathbf{v}}$ (defined in (1.1)) in the hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2$. Let

$$(1.2) \quad O = \mathbb{P} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

be the *origin* in $\mathbb{H}_{\mathbb{R}}^2$. Although we will not need this, the hyperbolicity relates to other more familiar quantities. For example, the hyperbolicity of a vector \mathbf{v} relates to the distance from $l_{\mathbf{v}}$ to the origin O in $\mathbb{H}_{\mathbb{R}}^2$ and to the Euclidean length of \mathbf{v} by:

$$\begin{aligned} \rho(\mathbf{x}^+(\mathbf{v}), \mathbf{x}^-(\mathbf{v})) &= 2 \operatorname{sech}(d(O, l_{\mathbf{v}})) \\ &= 2 \sqrt{\frac{2}{1 + \|\mathbf{v}\|^2}}. \end{aligned}$$

The set of all ϵ -spacelike vectors is the compact set

$$\mathfrak{S}_{\epsilon} = \mathfrak{S} \cap B(0, \sqrt{8/\epsilon^2 - 1})$$

and $\mathfrak{S} = \bigcup_{\epsilon > 0} \mathfrak{S}_{\epsilon}$.

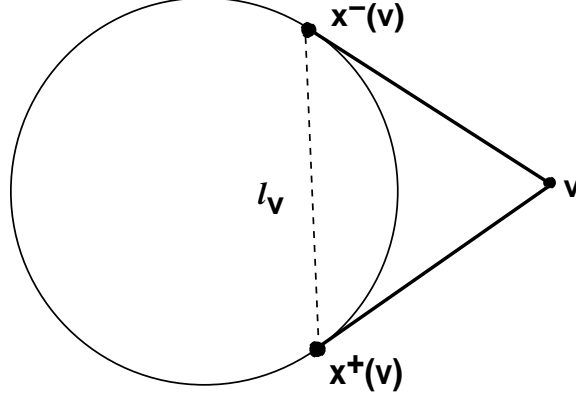


FIGURE 1. A null frame

1.5. Compression. We now prove the basic technical lemma bounding the compression of Euclidean balls by isometries of \mathbb{E} . A similar bound was found in Lemma 3 (pp. 686–687) of Drumm [5] (see also (1) of §3.7 of Drumm [4]).

Definition 1.5. For any unit-spacelike vector \mathbf{v} , the weak-unstable plane $E^{wu}(\mathbf{v}) \subset \mathbb{R}^{2,1}$ is the plane spanned by the vectors \mathbf{v} and $\mathbf{x}^+(\mathbf{v})$. If $g \in \mathrm{SO}(2, 1)$ is hyperbolic, then $E^{wu}(g)$ is defined as $E^{wu}(\mathbf{x}^0(g))$. If $x \in \mathbb{E}$, then $E_x^{wu}(g)$ is defined as the image of $E^{wu}(g)$ under translation by x .

(Drumm [3, 4, 5] denotes E^{wu} by S_+ .)

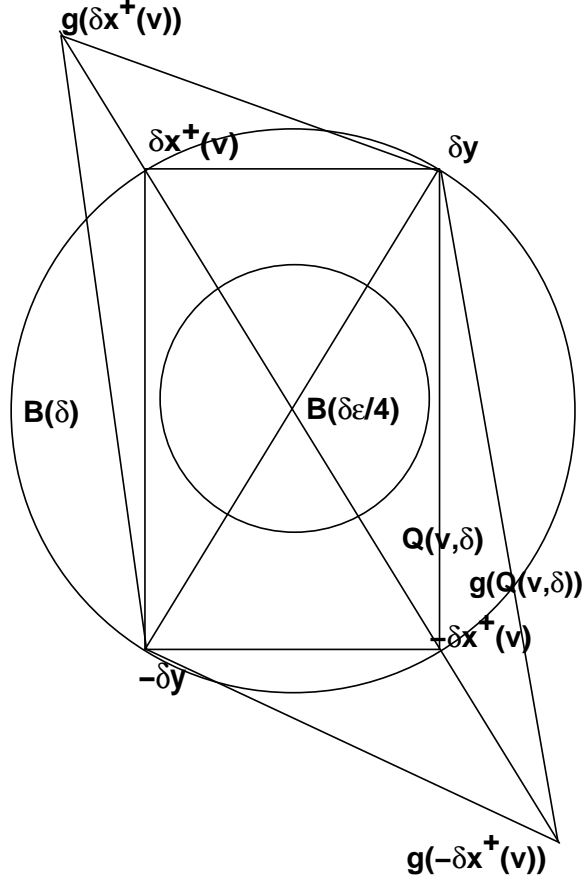


FIGURE 2. The Compression Lemma

Lemma 1.6. *Let \mathbf{v} be an ϵ -spacelike vector. Let $\delta > 0$ and $Q(\mathbf{v}, \delta)$ be the convex hull of the four intersection points of $\partial B(0, \delta)$ with the two lines $\mathbb{R}\mathbf{v}$ and $\mathbb{R}\mathbf{x}^+(\mathbf{v})$. Then $Q(\mathbf{v}, \delta)$ is a rectangle in the plane $E^{wu}(\mathbf{v})$ containing*

$$B\left(0, \frac{\delta\epsilon}{4}\right) \cap E^{wu}(\mathbf{v}).$$

PROOF. The lines $\mathbb{R}\mathbf{v}$ and $\mathbb{R}\mathbf{x}^+(\mathbf{v})$ meet $\partial B(0, \delta)$ in the four points $\pm\delta\mathbf{x}^+(\mathbf{v}), \pm\delta\mathbf{y}$ where $\mathbf{y} = \mathbf{v}/\|\mathbf{v}\|$. (Compare Figure 2.) Since $\triangle(\mathbf{x}^+(\mathbf{v}), \mathbf{y}, \mathbf{x}^-(\mathbf{v}))$ is isosceles,

$$\rho(\mathbf{x}^+(\mathbf{v}), \mathbf{y}) = \rho(\mathbf{x}^-(\mathbf{v}), \mathbf{y}),$$

the triangle inequality implies

$$\begin{aligned} \epsilon &\leq \rho(\mathbf{x}^+(\mathbf{v}), \mathbf{x}^-(\mathbf{v})) \\ &\leq \rho(\mathbf{x}^+(\mathbf{v}), \mathbf{y}) + \rho(\mathbf{y}, \mathbf{x}^-(\mathbf{v})) \\ &\leq 2\rho(\mathbf{x}^+(\mathbf{v}), \mathbf{y}). \end{aligned}$$

Therefore $\rho(\mathbf{x}^+(\mathbf{v}), \mathbf{y}) \geq \epsilon/2$. Similarly $\rho(\mathbf{x}^+(\mathbf{v}), -\mathbf{y}) \geq \epsilon/2$. Thus the sides of $Q(\mathbf{v}, \delta)$ have length at least $\delta\epsilon/2$, and $B(0, \delta\epsilon/4) \subset Q(\mathbf{v}, \delta)$ as claimed. \square

Lemma 1.7. *Suppose that $g \in \mathrm{SO}^0(2, 1)$ is ϵ -hyperbolic. Then for all $\delta > 0$,*

$$B\left(0, \frac{\delta\epsilon}{4}\right) \cap E^{wu}(g) \subset gB(0, \delta).$$

PROOF. g fixes $\pm\delta y$ and multiplies $\pm\delta x^+(g)$ by $\lambda(g)^{-1} > 1$ so $Q \subset g(Q)$. (Compare Figure 2.) By Lemma 1.6, $B(0, \delta\epsilon/4) \cap E^{wu}(g) \subset Q \subset gB(0, \delta)$. \square

The following lemma directly follows from Lemma 1.7 by applying translations.

Lemma 1.8 (The Compression Lemma). *Suppose that $h \in \mathrm{Isom}^0(\mathbb{E})$ is an ϵ -hyperbolic affine isometry. Then for all $\delta > 0$ and $x \in \mathbb{E}$,*

$$B\left(h(x), \frac{\delta\epsilon}{4}\right) \cap E_{h(x)}^{wu}(g) \subset h(B(x, \delta)).$$

2. Schottky groups

In this section we recall the construction of Schottky subgroups in $\mathrm{SO}^0(2, 1)$ and their action on $\mathbb{H}_{\mathbb{R}}^2$. We also use the projective action $\mathbb{P}(g)$ of $g \in \mathrm{SO}^0(2, 1)$ on S^1 (see §1.2) and abbreviate $\mathbb{P}(g)$ by g to ease notation.

This classical construction is the template for the construction of affine Schottky groups later in §3.4. Then we prove several elementary technical facts to be used later in the proof of the Main Theorem.

2.1. Schottky's configuration. Let $G \subset \mathrm{SO}^0(2, 1)$ be a group generated by g_1, \dots, g_m , for which there exist intervals $A_i^-, A_i^+ \subset S^1$, $i = 1, \dots, m$ such that:

- $g_i(A_i^-) = \partial\mathcal{N}_+ - \bar{A}_i^+$;
- $g_i^{-1}(A_i^+) = \partial\mathcal{N}_+ - \bar{A}_i^-$.

We call G a *Schottky group*. Write J for the set $\{+1, -1\}$ or its abbreviated version $\{+, -\}$. Denote by I the set $\{1, \dots, m\}$. We index many of the objects associated with Schottky groups (for example the intervals A_i^j and the Schottky generators g_i^j) by the Cartesian product

$$I \times J = \{1, \dots, m\} \times \{+, -\}.$$

Let H_i^+ and H_i^- be the two half-spaces (the convex hulls) in $\mathbb{H}_{\mathbb{R}}^2$ bounded by A_i^+ and A_i^- respectively. Their complement

$$\Delta_i = \mathbb{H}_{\mathbb{R}}^2 - (\bar{H}_i^+ \cup \bar{H}_i^-)$$

is the convex hull in $\mathbb{H}_{\mathbb{R}}^2$ of $\partial\mathbb{H}_{\mathbb{R}}^2 - (\bar{A}_i^+ \cup \bar{A}_i^-)$. These half-spaces satisfy conditions analogous to those above :

- $g_i(H_i^-) = \mathbb{H}_{\mathbb{R}}^2 - \bar{H}_i^+$;
- $g_i^{-1}(H_i^+) = \mathbb{H}_{\mathbb{R}}^2 - \bar{H}_i^-$.

(Compare Fig. 3.) Δ_i is a fundamental domain for the cyclic group $\langle g_i \rangle$. As all of the A_i^j are disjoint, all of the complements $\mathbb{H}_{\mathbb{R}}^2 - \Delta_i$ are disjoint.

Lemma 2.1. *For each $i = 1, \dots, m$, $x^+(g_i) \in A_i^+$ and $x^-(g_i) \in A_i^-$.*

PROOF. Since g_i is hyperbolic, it has three invariant lines corresponding to its eigenvalues. The two eigenvectors corresponding to λ and λ^{-1} are null, determining exactly two fixed points of g_i on S^1 . The fixed point corresponding to $x^+(g_i)$ is attracting and the fixed point corresponding to $x^-(g_i)$ is repelling. Since $A_i^+ \subset S^1 - A_i^-$,

$$g_i(A_i^+) \subset g_i(S^1 - A_i^-) = A_i^+$$

so Brouwer's fixed-point theorem implies that either $x^+(g_i)$ or $x^-(g_i)$ lies in A_i^+ . The same argument applied to g_i^{-1} implies that either $x^+(g_i)$ or $x^-(g_i)$ lies in A_i^- . Since g has only two fixed points and $A_i^- \cap A_i^+ = \emptyset$, either $x^+(g_i) \in A_i^+, x^-(g_i) \in A_i^-$ or $x^+(g_i) \in A_i^-, x^-(g_i) \in A_i^+$. The latter case cannot happen since $x^+(g_i)$ is attracting and $x^-(g_i)$ is repelling. \square

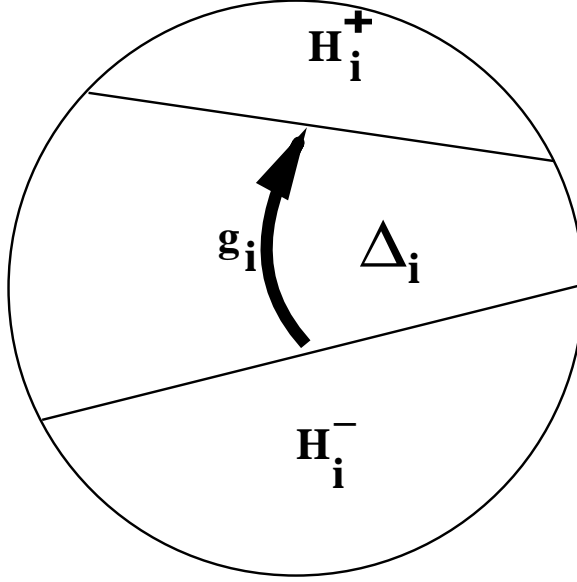


FIGURE 3. Half-planes defining a fundamental domain for a cyclic hyperbolic group

The following theorem is the basic result on Schottky groups. It is one of the simplest cases of “Poincaré’s theorem on fundamental polygons” or “Klein’s combination theorem.” Compare Beardon [1], Ford [11], Ratcliffe [17], pp. 584–587, and Epstein-Petronio [10].

Theorem 2.2. *The set $\{g_1, \dots, g_m\}$ freely generates G and G is discrete. The intersection $\Delta = \Delta_1 \cap \dots \cap \Delta_m$ is a fundamental domain for G acting on $H_{\mathbb{R}}^2$.*

Figure 4 depicts the pattern of identifications.

We break the proof into three separate lemmas: first, that the $g\Delta$ form a set of disjoint tiles, second, that G is discrete and third, that these tiles cover all of $H_{\mathbb{R}}^2$. The first lemma extends immediately to affine Schottky groups. The second lemma is automatic since the linear part of Γ equals G , which we already know is discrete. However, a much different argument is needed to prove that the tiles cover \mathbb{E} in the affine case.

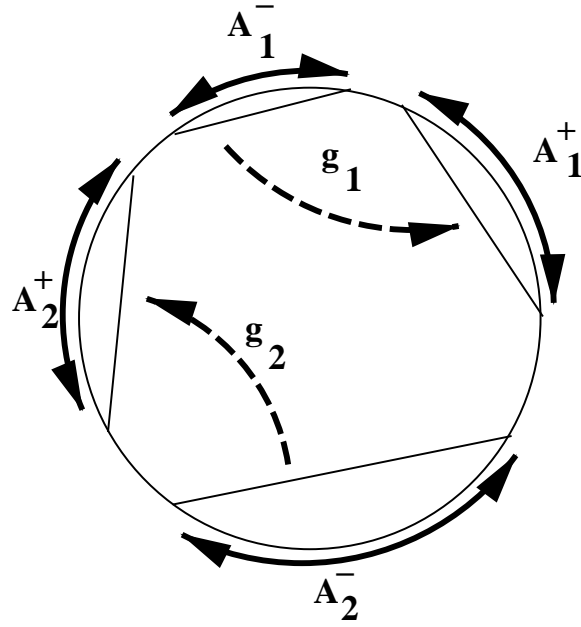


FIGURE 4. Generators for a Schottky group

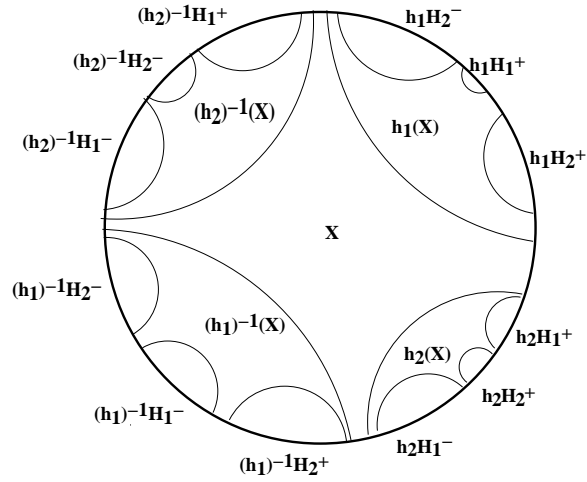


FIGURE 5. The tiling associated to a Schottky group, in the Poincaré disk model

Lemma 2.3. *If $g \in G$ is nontrivial, then $g\Delta \cap \Delta = \emptyset$.*

Lemma 2.4. *G is discrete.*

Lemma 2.5. *The union*

$$G\bar{\Delta} = \bigcup_{g \in G} g(\bar{\Delta})$$

equals all of $H_{\mathbb{R}}^2$.

PROOF OF LEMMA 2.3. We show that if $g \in G$ is a reduced word

$$g = g_{i_k}^{j_k} \dots g_{i_1}^{j_1}$$

(where $j_i = \pm 1$) then either $k = 0$ (that is, $g = 1$) or $g\Delta \cap \Delta = \emptyset$. In the latter case $g\bar{\Delta} \cap \bar{\Delta} = \emptyset$ unless $k = 1$. This implies that the g_i freely generate G and that G acts properly and freely on the union

$$\Gamma\bar{\Delta} = \bigcup_{\gamma \in \Gamma} \gamma\bar{\Delta}.$$

Then we show that $\Gamma\bar{\Delta} = H_{\mathbb{R}}^2$.

Suppose that $k > 0$. We claim inductively that $g(\Delta) \subset H_{i_k}^{j_k}$. Let $g' = g_{i_{k-1}}^{j_{k-1}} \dots g_{i_1}^{j_1}$ so that $g = g_{i_k}^{j_k} g'$. If $k = 1$, then $g' = 1$ and

$$g\Delta = g_{i_1}^{j_1} \Delta \subset g_{i_1}^{j_1} (H_{\mathbb{R}}^2 - \bar{H}_{i_1}^{-j_1}) \subset H_{i_1}^{j_1} \subset H_{\mathbb{R}}^2 - \Delta.$$

If $k > 1$, then $g_{i_k}^{-j_k} \neq g_{i_{k-1}}^{j_{k-1}}$ (since g is a reduced word) so $H_{i_k}^{-j_k} \neq H_{i_{k-1}}^{j_{k-1}}$. The induction hypothesis

$$g'(\Delta) \subset H_{i_{k-1}}^{j_{k-1}}$$

implies

$$g\Delta = g_{i_k}^{j_k} g'(\Delta) \subset g_{i_k}^{j_k} H_{i_{k-1}}^{j_{k-1}} \subset g_{i_k}^{j_k} (H_{\mathbb{R}}^2 - \bar{H}_{i_k}^{-j_k}) \subset H_{i_k}^{j_k}$$

as desired. Thus all of the $\gamma\Delta$, for $\gamma \in \Gamma$, are disjoint and their closures $\gamma\bar{\Delta}$ tile $\Gamma\bar{\Delta}$. \square

PROOF OF LEMMA 2.4. Let $x_0 \in \Delta$. Since Δ is open, there exists $\delta_0 > 0$ such that the δ_0 -ball (in the hyperbolic metric d on $H_{\mathbb{R}}^2$) about x_0 lies in Δ . We have proved that if g is a reduced word in g_1, \dots, g_m then $g(x_0) \notin \Delta$ so $d(x_0, g(x_0)) > \delta_0$. In particular no sequence of reduced words in G can accumulate on the identity, proving that G is discrete. \square

PROOF OF LEMMA 2.5. We must prove $G\bar{\Delta} = H_{\mathbb{R}}^2$. We use a completeness argument to show that the quotient $M = (G\bar{\Delta})/G$ is actually $H_{\mathbb{R}}^2/G$. We begin by making an abstract model for the universal covering space \tilde{M} as the quotient space of the Cartesian product $\bar{\Delta} \times G$ by the equivalence relation generated by identifications

$$(x, g) \sim (g_i^j x, gg_i^{-j})$$

where $x \in \partial H_i^{-j}$, $g \in G$ and $(i, j) \in I \times J$. Then $M = (G\bar{\Delta})/G$ inherits a hyperbolic structure from $H_{\mathbb{R}}^2$. We show that this hyperbolic structure is complete to prove that $M = H_{\mathbb{R}}^2/G$ and thus $B\bar{\Delta} = H_{\mathbb{R}}^2$.

The identifications define the structure of a smooth manifold on \tilde{M} . If $x \in \Delta$, then the equivalence class of (x, g) contains only (x, g) itself, and a smooth chart at (x, g) arises from the smooth structure on Δ . If $x \in \partial\Delta$, then the equivalence class of (x, g) equals

$$\{(x, g), (g_i^j x, gg_i^{-j})\}$$

where $x \in \partial H_i^{-j}$. Let U be an open neighborhood of x in $H_{\mathbb{R}}^2$ which intersects the orbit Γx in $\{x\}$. Since x is a boundary point of the smooth surface-with-boundary

$\bar{\Delta}$, the intersection $U \cap \bar{\Delta}$ is a coordinate patch for x in $\bar{\Delta}$. The image $g_i^j(U - \Delta)$ is a smooth coordinate patch for $g_i^j x$ in $\bar{\Delta}$ and the image of the union

$$(U \cap \bar{\Delta}) \times \{g\} \cup (g_i^j(U - \Delta)) \times \{gg_i^{-j}\}$$

is a smooth coordinate patch for the equivalence class of (x, g) .

The G -action defined by

$$\gamma : (x, g) \mapsto (x, \gamma g)$$

preserves this equivalence relation and thus defines a G -action on the quotient \tilde{M} . The map

$$\begin{aligned} D : \bar{\Delta} \times G &\longrightarrow \mathbb{H}_{\mathbb{R}}^2 \\ (x, g) &\longmapsto g(x) \end{aligned}$$

preserves the equivalence relation and defines a G -equivariant map, the *developing map* $D : \tilde{M} \longrightarrow \mathbb{H}_{\mathbb{R}}^2$. The developing map D is a local diffeomorphism onto the open set $G\bar{\Delta}$.

Pull back the hyperbolic metric from $\mathbb{H}_{\mathbb{R}}^2$ by D to obtain a Riemannian metric on \tilde{M} for which D is a local isometry. Since G acts isometrically on $\mathbb{H}_{\mathbb{R}}^2$, the developing map D is G -equivariant. We claim \tilde{M} is geodesically complete. To this end, consider a maximal unit-speed geodesic ray $\mu(t)$ defined for $0 < t < t_0$. Its preimage in $\bar{\Delta} \times G$ consists of geodesic segments μ_k in various components $\bar{\Delta} \times \{g_{i_k}\}$.

We claim there are only finitely many segments μ_k . Since $\bar{\Delta}$ is convex, one of its segments μ_k cannot enter and exit $\bar{\Delta}$ from the same side. Since the defining arcs A_i^j are pairwise disjoint, the corresponding geodesics ∂H_i^j are pairwise ultraparallel and the distance between different sides of $\bar{\Delta}$ is bounded below by $\delta > 0$. Since the length of μ equals t_0 , there can be at most t_0/δ segments μ_k .

Let $\mu_k : [t_1, t_0) \longrightarrow \bar{\Delta} \times \{g_{i_k}\}$ be the last geodesic segment. Since $\bar{\Delta}$ is closed, $\mu_k(t)$ converges as $t \longrightarrow t_0$, contradicting maximality.

Thus \tilde{M} is geodesically complete. Since a local isometry from a complete Riemannian manifold is a covering space, D is a covering space. The van Kampen theorems imply that \tilde{M} is simply connected, so that D is a diffeomorphism and hence surjective. Thus $G\bar{\Delta} = \mathbb{H}_{\mathbb{R}}^2$ as desired. \square

2.2. Existence of a small interval. The following lemma is an elementary fact which is used in the proof of completeness. We assume that the number m of generators in the Schottky group is at least 2.

Lemma 2.6. *Let $\{A_i^j \mid (i, j) \in I \times J\}$ be a collection of disjoint intervals on $S^2 \cap \mathcal{N}_+$. Then there exists an $(i_0, j_0) \in I \times J$ such that the length $\Phi(A_{i_0}^{j_0}) < \pi/2$.*

PROOF. Since the A_i^j are disjoint, their total length is bounded by 2π . Since there are $2m \geq 4$ of them, at least one has length $< (2\pi)/4$ as claimed. \square

2.3. A criterion for ϵ -hyperbolicity. To determine proper discontinuity of affine Schottky groups, we examine sequences of group elements. An important case is when for some $\epsilon > 0$, every element of the sequence is ϵ -hyperbolic. Here is a useful criterion for such ϵ -hyperbolicity of an entire sequence. As before, $A_i^j, (i, j) \in I \times J$, denote the disjoint intervals associated to the generators g_1, \dots, g_m of a Schottky group.

Lemma 2.7. *Let θ_0 be the minimum angular separation between the intervals $A_i^j \subset S^1$ and let $\epsilon_0 = 2 \sin(\theta_0/2)$. Suppose that*

$$(2.1) \quad g = g_{i_0}^{j_0} \cdots g_{i_l}^{j_l}$$

is a reduced word. If $(i_l, j_l) \neq (i_0, -j_0)$ then g is ϵ_0 -hyperbolic.

The condition $(i_l, j_l) \neq (i_0, -j_0)$ means that (2.1) describes a *cyclically reduced* word.

PROOF. By Lemma 2.1,

$$x^+(g) \in A_{i_0}^{j_0}$$

and

$$x^-(g) = x^+(g^{-1}) \in A_{i_l}^{-j_l}.$$

Since $(i_l, j_l) \neq (i_0, -j_0)$, the vectors $x^-(g)$ and $x^+(g)$ lie in the attracting interval and repelling interval respectively. Since they lie in disjoint conical neighborhoods, they are separated by at least θ_0 . \square

This lemma is crucial in the analysis of sequences of elements of Γ arising from incompleteness. These sequences will all be ϵ -hyperbolic for some $\epsilon > 0$. A typical sequence which is not ϵ -hyperbolic for any $\epsilon > 0$ is the following (compare [7]):

$$\gamma_n = g_1^n g_2 g_1^{-n}.$$

Since all the elements are conjugate, the eigenvalues are constant (in particular they are bounded). However both sequences of eigenvectors $x^-(\gamma_n), x^+(\gamma_n)$ converge to $x^+(g_1)$ so $\rho(x^-(\gamma_n), x^+(\gamma_n)) \rightarrow 0$.

3. Crooked planes and zigzags

Now consider the action of a Schottky group $G \subset \mathrm{SO}^0(2, 1)$ on Minkowski $(2 + 1)$ -space \mathbb{E} . The inverse projectivization $\mathbb{P}^{-1}(\Delta)$ of a fundamental domain Δ is a fundamental domain for the linear action of G on the subspace \mathcal{N} of timelike vectors. We extend this fundamental domain to a larger open subset of \mathbb{E} . The extended fundamental domains are bounded by polyhedral surfaces called *crooked planes*. For the groups of linear transformations, the crooked planes all pass through the origin — indeed the origin is a special point of each crooked plane, its *vertex*.

The Schottky group $G = \langle g_1, \dots, g_m \rangle$ acts properly discontinuously and freely on the open subset \mathcal{N} consisting of timelike vectors in $\mathbb{R}^{2,1}$. However, since G fixes the origin, the G -action on all of \mathbb{E} is quite far from being properly discontinuous.

Then we deform G inside the group of affine isometries of \mathbb{E} to obtain a group Γ which in certain cases acts freely and properly discontinuously. This *affine deformation* Γ of G is defined by geometric identifications of a family of *disjoint* crooked planes. The crooked planes bound *crooked half-spaces* whose intersection is a *crooked polyhedron* X . In [4], Drumm proved the remarkable fact that as long as the crooked planes are disjoint, Γ acts freely and properly discontinuously on \mathbb{E} with fundamental domain X .

3.1. Extending Schottky groups to Minkowski space. When Δ is a fundamental domain for G acting on $\mathbb{H}_{\mathbb{R}}^2$, its inverse image $\mathbb{P}^{-1}(\Delta)$ is a fundamental domain for the action of G on \mathcal{N} . The faces of $\mathbb{P}^{-1}(\Delta)$ are the intersections of \mathcal{N} with indefinite planes corresponding to the geodesics in $\mathbb{H}_{\mathbb{R}}^2$ forming the sides of Δ .

Each face S of $\mathbb{P}^{-1}(\Delta)$ extends to a polyhedral surface \mathcal{C} in $\mathbb{R}^{2,1}$ called a *crooked plane*. The face $S \subset \mathbb{P}^{-1}(\Delta)$ is the *stem* of the crooked plane. To extend the stem one adds two null half-planes, called the *wings*, along the null lines bounding S . Crooked planes are more flexible than Euclidean planes since one can build fundamental polyhedra for free, properly discontinuous groups from them.

Crooked planes were introduced by Drumm in his doctoral dissertation [3] (see also [4, 5]). Their geometry is extensively discussed in Drumm-Goldman [8] (see also [9]) where their intersections are classified.

3.2. Construction of a crooked plane. Here is an example from which we derive the general definition of a crooked plane. The geodesic l_v determined by the spacelike vector

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

corresponds to the set

$$S_0 = \overline{(\mathbb{P}^{-1}(l_v))} = \left\{ \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} \mid |u_3| \geq |u_2| \right\} \subset \mathcal{N}$$

which is the *stem* of the crooked plane. The two lines bounding S_0 are

$$\begin{aligned} \partial^- S_0 &= \left\{ \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} \mid u_2 = u_3 \right\} \\ \partial^+ S_0 &= \left\{ \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} \mid u_2 = -u_3 \right\} \end{aligned}$$

and the *wings* are the half-planes

$$\begin{aligned} \mathcal{W}_0^- &= \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mid u_1 \leq 0, u_2 = u_3 \right\} \\ \mathcal{W}_0^+ &= \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mid u_1 \geq 0, u_2 = -u_3 \right\}. \end{aligned}$$

The crooked plane \mathcal{C}_0 is defined as the union

$$\mathcal{C}_0 = \mathcal{W}_0^- \cup S_0 \cup \mathcal{W}_0^+.$$

(Compare Figure 6.) Corresponding to the half-plane $H_v \subset \mathbb{H}_{\mathbb{R}}^2$ is the region

$$\tilde{H}_v = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathcal{N} \mid u_1 > 0, u_3 > 0 \right\}$$

and the component of the complement $\mathbb{E} - \mathcal{C}_0$ containing \tilde{H}_v is the *crooked half-space*

$$\begin{aligned} \mathcal{H}(v) = & \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mid u_1 > 0, u_2 + u_3 > 0 \right\} \\ & \cup \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mid u_1 = 0, u_2 + u_3 > -0, u_2 + u_3 > 0 \right\} \\ & \cup \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mid u_1 < 0, -u_2 + u_3 > 0 \right\}. \end{aligned}$$

Now let $u \in \mathfrak{S}$ be any unit-spacelike vector, determining the half-space $H_u \in \mathbb{H}_{\mathbb{R}}^2$. The crooked plane directed by u can be defined as follows, using the previous example. Let $g \in \text{SO}^0(2, 1)$ such that $g(v) = u$. The *crooked plane directed by u* is $\mathcal{C}(u) = g(\mathcal{C}_0)$.

Since g preserves the spacelike, lightlike or timelike nature of a vector, we see that $\mathcal{C}(u)$ is composed of a stem flanked by two tangent wings, just like \mathcal{C}_0 .

The crooked plane is singular at the origin, which we call the *vertex* of $\mathcal{C}(u)$. In general, if $p \in \mathbb{E}$ is an arbitrary point, the *crooked plane directed by u and with vertex p* is defined as:

$$\mathcal{C}(u, p) = p + \mathcal{C}(u).$$

Let $\mathcal{C}(u, p) \subset \mathbb{E}$ be a crooked plane.

We define the crooked half-space $\mathcal{H}(u, p)$ to be the component of the complement $\mathbb{E} - \mathcal{C}(u, p)$ which is bounded by $\mathcal{C}(u, p)$ and contains $p + H_u$. Note that \mathbb{E} decomposes as a disjoint union

$$\mathbb{E} = \mathcal{H}(u, p) \cup \mathcal{C}(u, p) \cup \mathcal{H}(-u, p).$$

(Crooked half-spaces are called *wedges* in Drumm [3, 4, 5].)

The *angle* $\Phi(\mathcal{H}(u, p))$ of the crooked half-space $\mathcal{H}(u, p)$ is taken to be the angle of A , the interval determined by the half-space H_u .

3.3. Zigzags. To understand the tiling of \mathbb{E} by crooked polyhedra, we intersect the tiling with a fixed definite plane P , which is always transverse to the stem and wings of any crooked plane. Since the tiling only contains countably many crooked planes, we may assume that P misses the vertices of the crooked planes in the tiling.

A *zigzag* in a definite plane P is a union ζ of two disjoint rays r_0 and r_1 and the segment s (called the *stem*) joining the endpoint v_0 of r_0 to the endpoint v_1 of r_1 , such that the two angles θ_0 and θ_1 formed by the rays at the respective endpoints of s differ by π radians. The intersection of a crooked plane with a definite plane not containing its vertex is a zigzag. (Conversely, every zigzag extends to a unique crooked plane, although we do not need this fact.) Compare Figure 7.

A *zigzag region* is a component \mathcal{Z} of $P - \zeta$ where $\zeta \subset P$ is a zigzag. Equivalently \mathcal{Z} is the intersection of a crooked half-space with P . The corresponding half-space in $\mathbb{H}_{\mathbb{R}}^2$ is bounded by an interval $A \subset S^1$ whose length $\Phi(A)$ is defined as the *angle* $\Phi(\mathcal{Z})$ of the zigzag region. One of the angles of \mathcal{Z} equals $\Phi(A)/2$ and the other, $\Phi(A)/2 + \pi$. Compare Figure 8.

If $r \subset P$ is an open ray contained in the zigzag region \mathcal{Z} , then r lies in a unique maximal open ray inside \mathcal{Z} (just move the endpoint of r back until it meets ζ).

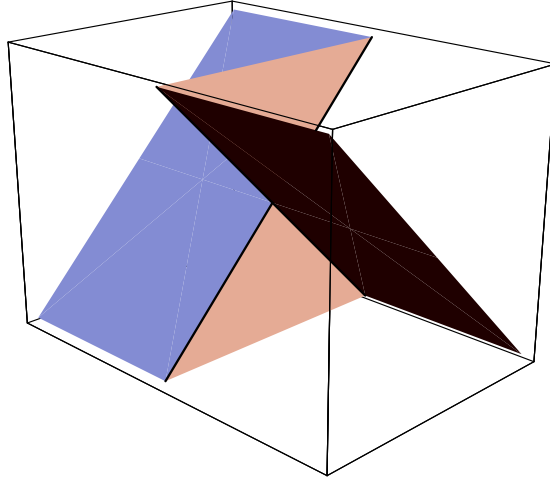
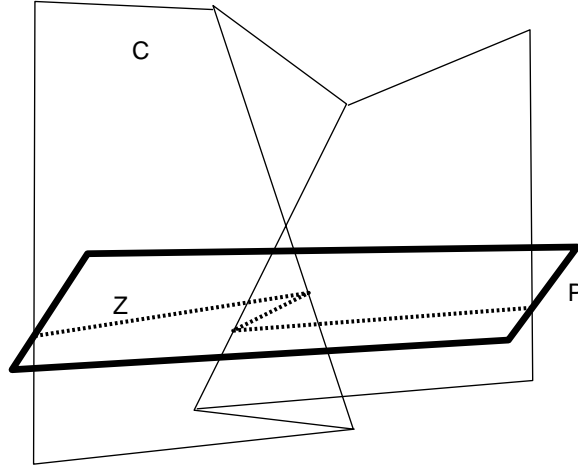


FIGURE 6. A crooked plane

FIGURE 7. The intersection of a crooked plane C with a definite plane P is a zigzag Z .

Every maximal open ray then lies in one of two angular sectors (possibly both). One angular sector has vertex v_0 and sides r_0 and s , and subtends the angle θ_0 . The other angular sector has vertex v_1 and sides r_1 and s , and subtends the angle θ_1 . Any two rays in R make an angle of at most $\Phi(\zeta)$ with each other.

3.4. Affine deformations. Consider a group Γ generated by isometries

$$h_1, \dots, h_m \in \text{Isom}^0(\mathbb{E})$$

for which there exist crooked half-spaces \mathcal{H}_i^j , where $(i, j) \in I \times J$ (using the indexing convention defined in §2) such that:

- all the \mathcal{H}_i^j are pairwise disjoint;

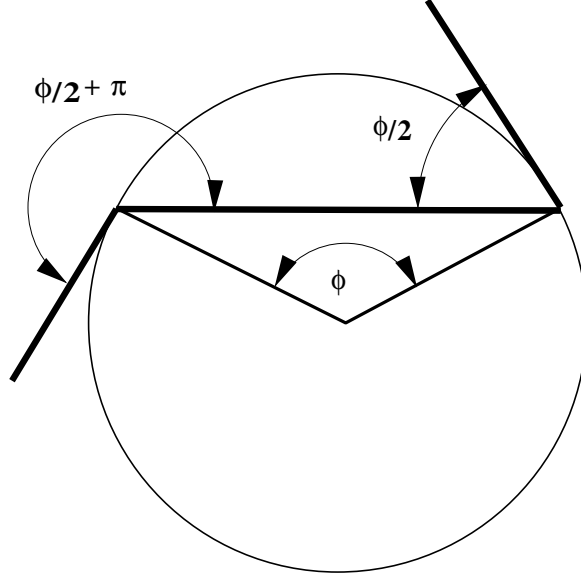


FIGURE 8. Angles in a zigzag

- $h_i(\mathcal{H}_i^-) = \mathbb{E} - \bar{\mathcal{H}}_i^+$. (Thus $h_i^{-1}(\mathcal{H}_i^+) = \mathbb{E} - \bar{\mathcal{H}}_i^-$ as well.)

We call Γ an *affine Schottky group*. The set

$$X = \mathbb{E} - \bigcup_{(i,j) \in I \times J} \bar{\mathcal{H}}_i^j$$

is an open subset of \mathbb{E} whose closure \bar{X} is a finite-sided polyhedron in \mathbb{E} bounded by the crooked planes $\mathcal{C}_i^j = \partial \mathcal{H}_i^j$.

Drumm-Goldman [8] provides criteria for disjointness of crooked half-spaces. In particular for every configuration of disjoint half-planes

$$H_1^+, H_1^-, \dots, H_m^+, H_m^- \subset \mathbb{H}_{\mathbb{R}}^2$$

paired by Schottky generators $g_1, \dots, g_m \in \mathrm{SO}^0(2, 1)$, there exists a configuration of disjoint crooked half-spaces

$$\mathcal{H}_1^+, \mathcal{H}_1^-, \dots, \mathcal{H}_m^+, \mathcal{H}_m^- \subset \mathbb{E}$$

whose directions correspond to the H_i^j , and which are paired by affine transformations h_i with linear part g_i , satisfying the above conditions.

We show that \bar{X} is a fundamental polyhedron for Γ acting on \mathbb{E} . As with standard Schottky groups, one first shows that the images $h\bar{X}$ form a set of *disjoint* tiles of $\Gamma\bar{X}$.

Lemma 3.1. $\Gamma\bar{X}$ is open.

PROOF. (Compare the proof of Lemma 2.5.) If $x \in X$, then γx is an interior point of $\Gamma X \subset \Gamma\bar{X}$ for every $\gamma \in \Gamma$. Otherwise suppose $x \in \partial X$. Then $x \in \mathcal{C}_i^j$ for some $(i, j) \in I \times J$. Let B be an open ball about x such that $B \cap \partial X \subset \mathcal{C}_i^j$. Then

$$(B \cap \bar{X}) \cup (h_i^{-j} B \cap \bar{X}) \subset \bar{X}$$

is an open subset of \bar{X} whose orbit is an open neighborhood of x in $\Gamma\bar{X}$. \square

The analogue of Lemma 2.3 is:

Lemma 3.2. *The affine transformations h_1, \dots, h_m freely generate Γ . The crooked polyhedron \bar{X} is a fundamental domain for Γ acting on $\Gamma\bar{X}$.*

PROOF. The proof is completely identical to that of Lemma 2.3. Replace the hyperbolic half-spaces H_i^j by crooked half-spaces \mathcal{H}_i^j , the hyperbolic polygon Δ by the crooked polyhedron X and the hyperbolic isometries g_i by Lorentzian affine isometries h_i . \square

The most difficult part of the proof of the Main Theorem is that $\Gamma\bar{X} = \mathbb{E}$, that is, the images $\gamma\bar{X}$ tile *all* of \mathbb{E} . Due to the absence of an invariant *Riemannian* metric, the completeness proof of Lemma 2.5 fails.

4. Completeness

We prove that the images of the crooked polyhedron X tile \mathbb{E} . We suppose there exists a point p not in $\Gamma\bar{X}$ and derive a contradiction.

The first step is to describe a sequence of nested crooked half-spaces \mathfrak{H}_k containing p . This sequence corresponds to a sequence of indices

$$(i_0, j_0), (i_1, j_1), \dots, (i_k, j_k), \dots$$

such that $\mathfrak{H}_k = \gamma_k \mathcal{H}_{i_k}^{j_k}$ where $\gamma_k = h_{i_0}^{j_0} \dots h_{i_{k-1}}^{j_{k-1}}$.

Since crooked polyhedra are somewhat complicated and the elements of Γ exhibit different dynamical behavior in different directions, bounding the separation of the crooked polyhedra requires some care. To simplify the discussion we intersect this sequence with a fixed definite plane P so that the crooked half-spaces \mathfrak{H}_k intersect P in a sequence of nested zigzag regions containing p . We then approximate the zigzag regions by half-planes $\Pi_k \subset P$ (compare Figures 9 and 10) and show that for infinitely many k , the distance between the successive lines $L_k = \partial\Pi_k$ bounding Π_k is bounded below, to reach the contradiction. The Compression Lemma 1.7 gives a lower bound for $\rho(L_k, L_{k+1})$ whenever γ_k is ϵ -hyperbolic. Using the special form of the sequence γ_k and the Hyperbolicity Criterion (Lemma 2.7), we find infinitely many ϵ -hyperbolic γ_k for some $\epsilon > 0$ and achieve a contradiction.

4.1. Construction of the nested sequence. The complement $\mathbb{E} - \bar{X}$ consists of the $2m$ crooked half-spaces \mathcal{H}_i^j , which are bounded by crooked planes \mathcal{C}_i^j , indexed by $I \times J$.

Lemma 4.1. *Let $p \in \mathbb{E} - \Gamma\bar{X}$. There exists a sequence $\{\mathfrak{H}_k\}$ of crooked half-spaces such that*

- $\mathfrak{H}_k \supset \mathfrak{H}_{k+1}$ and $\mathfrak{H}_k \neq \mathfrak{H}_{k+1}$;
- $p \in \mathfrak{H}_k$;
- *there exists a sequence $(i_0, j_0), (i_1, j_1), \dots, (i_n, j_n), \dots$ in $I \times J$ such that $(i_k, j_k) \neq (i_{k+1}, -j_{k+1})$ for all $k \geq 0$ and*

$$\mathfrak{H}_k = \gamma_k \mathcal{H}_{i_k}^{j_k}$$

where

$$\gamma_k = h_{i_0}^{j_0} h_{i_1}^{j_1} \dots h_{i_{k-1}}^{j_{k-1}}.$$

PROOF. We first adjust p so that the first crooked half-space \mathfrak{H}_0 satisfies $\Phi(\mathfrak{H}_0) < \pi/2$. By Lemma 2.6,

$$(4.1) \quad \Phi(A_{i_0}^{j_0}) < \pi/2$$

for some $(i_0, j_0) \in I \times J$. Since $p \notin \bar{X}$, there exists (i, j) such that $p \in \mathcal{H}_i^j$. If $(i, j) \neq (i_0, j_0)$, then we replace p by γp , for some $\gamma \in \Gamma$ such that $\gamma p \in \mathcal{H}_{i_0}^{j_0}$. Here is how we do this. If $(i, j) \neq (i_0, -j_0)$, then $\gamma = h_{i_0}^{j_0}$ moves p into $Hh_{i_0}^{j_0}$. Otherwise first move p into a crooked half-space other than $\mathcal{H}_{i_0}^{-j_0}$, then into $\mathcal{H}_{i_0}^{j_0}$. For example, $h_{i_1}^{j_1}$ moves p into $\mathcal{H}_{i_1}^{j_1}$ and then $\gamma = h_{i_0}^{j_0} h_{i_1}^{j_1}$ moves p into $\mathcal{H}_{i_0}^{j_0}$. Thus we may assume that

$$p \in \mathfrak{H}_0 = \mathcal{H}_{i_0}^{j_0}$$

where $\Phi(\mathfrak{H}_0) < \pi/2$.

Suppose inductively that

$$\mathfrak{H}_0 \supset \cdots \supset \mathfrak{H}_k \ni p$$

is a nested sequence of crooked half-spaces containing p satisfying the conclusions of Lemma 4.1. Then $\mathfrak{H}_k = \gamma_k \mathcal{H}_{i_k}^{j_k}$ and $\gamma_k^{-1}(p) \in \mathcal{H}_{i_k}^{j_k}$. Let $\gamma_{k+1} = \gamma_k h_{i_k}^{j_k}$. Thus

$$\gamma_{k+1}^{-1}(p) \in h_{i_k}^{-j_k} \mathcal{H}_{i_k}^{j_k} = \mathbb{E} - \mathcal{H}_{i_k}^{-j_k}.$$

Since $p \notin \Gamma \bar{X}$,

$$\gamma_{k+1}^{-1}(p) \in \mathbb{E} - \bar{X} - \mathcal{H}_{i_k}^{-j_k} = \bigcup_{(i,j) \neq (i_k, -j_k)} \mathcal{H}_i^j.$$

Let (i_{k+1}, j_{k+1}) index the component of $\mathbb{E} - \bar{X} - \mathcal{H}_{i_k}^{-j_k}$ containing $\gamma_{k+1}^{-1}(p)$. This gives the desired sequence. \square

4.2. Uniform Euclidean width of crooked polyhedra. If $S \subset \Gamma \bar{X}$, define the *star-neighborhood* of S as the interior of the union of all tiles $\gamma \bar{X}$ intersecting \bar{S} .

Lemma 4.2. *There exists $\delta_0 > 0$ such that the δ_0 -neighborhood $B(X, \delta_0)$ lies in the star-neighborhood of \bar{X} . In particular whenever $(i, j), (i', j') \in I \times J$ satisfy $(i, j) \neq (i', -j')$,*

$$(4.2) \quad B(\mathbb{E} - \mathcal{H}_i^j, \delta_0) \subset \mathbb{E} - h_i^j \bar{\mathcal{H}}_{i'}^{j'}.$$

PROOF. The fundamental polyhedron \bar{X} is bounded by crooked planes $\mathcal{C}_i^j = \partial \mathcal{H}_i^j$. The star-neighborhood of \bar{X} equals

$$\bar{X} \cup \bigcup_{(i,j) \in I \times J} h_i^j \bar{X}.$$

Its complement consists of the $2m(2m-1)$ crooked half-spaces $h_i^j \mathcal{H}_{i'}^{j'}$ where $(i', j') \neq (i, -j)$. Unlike in hyperbolic space, two disjoint closed planar regions in Euclidean space are separated a positive distance apart. Four closed planar regions comprise a crooked plane. Thus the distance between disjoint crooked planes is strictly positive. Choose $\delta_0 > 0$ to be smaller than the distance between any of the \mathcal{C}_i^j and $h_i^j \mathcal{C}_{i'}^{j'}$. The second assertion follows since the δ_0 -neighborhood of a crooked half-space \mathfrak{H} equals $\mathfrak{H} \cup B(\partial \mathfrak{H}, \delta_0)$. \square

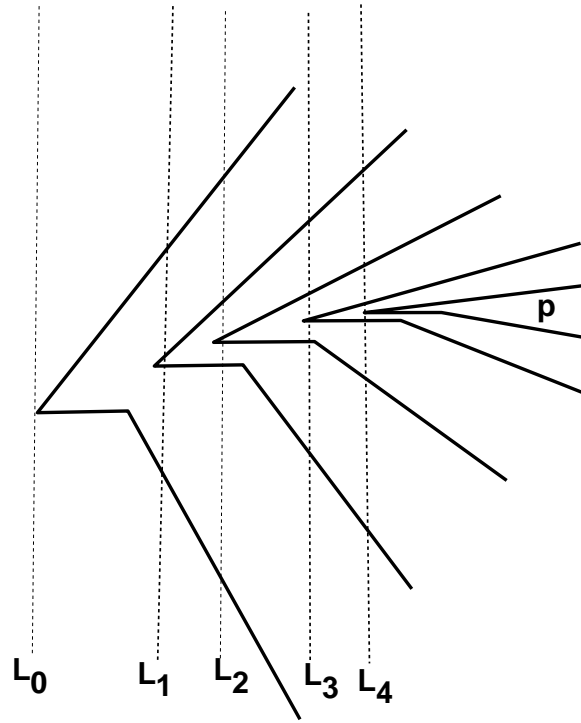


FIGURE 9. Accumulating zigzag regions

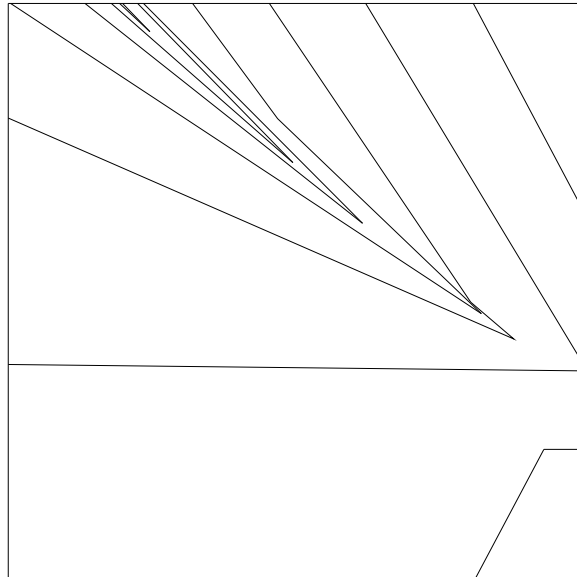


FIGURE 10. A sequence of zigzag regions tending to infinity

4.3. Approximating zigzag regions by half-planes. Now intersect with P . We approximate each zigzag region $\mathfrak{H}_k \cap P$ by a Euclidean half-plane $\Pi_k \subset P$ containing p . These half-planes form a nested sequence

$$\Pi_1 \supset \Pi_2 \supset \cdots \supset \Pi_k \supset \cdots$$

with $\mathfrak{H}_k \cap P \subset \Pi_k$ and we take the lines $L_k = \partial\Pi_k$ to be parallel. Since $p \in \mathfrak{H}_k \cap P$ for all k , and $p \in \partial(\Gamma\bar{X})$, the parallel lines L_k approach at p from one side. We obtain a contradiction by bounding the Euclidean distance $\rho(L_k, L_{k+1})$ from below, for infinitely many k .

Here is the detailed construction. Let ν be the line in P parallel to the intersection of P with the stem of $\partial\mathfrak{H}_0$. Then ν makes an angle of at most $\pi/4$ with every ray contained in $\mathfrak{H}_0 \cap P$. Let $L_k \subset P$ be the line perpendicular to ν bounding a half-plane Π_k containing $\mathfrak{H}_k \cap P$ and intersecting the zigzag $\zeta_k = \partial\mathfrak{H}_k \cap P$ at a vertex of ζ_k .

Choose $\delta_0 > 0$ as in Lemma 4.2.

Lemma 4.3. *For any $\delta \leq \delta_0$, the tubular neighborhood*

$$(4.3) \quad T_k(\delta) = \gamma_k(B(\gamma_k^{-1}L_k, \delta))$$

of L_k is disjoint from L_{k+1} .

PROOF. $L_k \subset P - \mathfrak{H}_k$ implies

$$(4.4) \quad \gamma_k^{-1}L_k \subset \mathbb{E} - \mathcal{H}_{i_k}^{j_k}.$$

Now

$$\begin{aligned} \mathbb{E} - \mathcal{H}_{i_k}^{j_k} &= \mathbb{E} - \gamma_k^{-1}\mathfrak{H}_k \\ &= \gamma_k^{-1}(\mathbb{E} - \mathfrak{H}_k) \\ &\subset \gamma_k^{-1}(\mathbb{E} - \mathfrak{H}_{k+1}) \\ &= \mathbb{E} - \gamma_k^{-1}\gamma_{k+1}\mathcal{H}_{i_{k+1}}^{j_{k+1}} \\ &= \mathbb{E} - h_{i_k}^{j_k}\mathcal{H}_{i_{k+1}}^{j_{k+1}}. \end{aligned}$$

Apply (4.4) and (4.2) to conclude $B(\gamma_k^{-1}, \delta) \subset \mathbb{E} - h_{i_k}^{j_k}\mathcal{H}_{i_{k+1}}^{j_{k+1}}$, so

$$\begin{aligned} T_k(\delta) &= \gamma_k B(\gamma_k^{-1}L_k, \delta) \\ &\subset \mathbb{E} - h_{i_0}^{j_0}h_{i_1}^{j_1} \cdots h_{i_{k-1}}^{j_{k-1}}(h_{i_k}^{j_k}\mathcal{H}_{i_{k+1}}^{j_{k+1}}) \\ &= E - \tilde{\mathfrak{H}}_{k+1}. \end{aligned}$$

so $T_k(\delta)$ is disjoint from $\tilde{\mathfrak{H}}_{k+1}$. Intersecting with P , $T_k(\delta)$ is disjoint from L_{k+1} . \square

4.4. Bounding the separation of half-planes. Write $E^k(x)$ for the weak-unstable plane $E_x^{wu}(g_k)$ where $x \in \mathbb{E}$ and g_k is the linear part of γ_k (see Definition 1.5). Foliate $T_k(\delta)$ by leaves $E^k(x) \cap T_k(\delta)$. We first bound the diameter of the leaves of $T_k(\delta)$.

Lemma 4.4. *The angle between ν and any $E^k(x) \cap P$ is bounded by $\pi/4$.*

PROOF. By Lemma 2.1, the vector $x^+(\gamma_k)$ lies in the attracting interval $A_{i_0}^{j_0}$. The null plane corresponding to $x^+(g_{i_0}^{j_0})$ is the weak-unstable plane $E^k(x)$. Since the stem of the crooked plane $\mathcal{C}_{i_0}^{j_0}$ corresponds to $A_{i_0}^{j_0}$, the corresponding null plane $E^k(x)$ intersects $\mathfrak{H}_0 = \mathcal{H}_{i_0}^{j_0}$ in a half-plane. (Compare Figure 11.)

Hence the line $E^k(x) \cap P$ meets $\mathfrak{H}_0 \cap P$ in a ray. Since any ray in $\mathfrak{H}_0 \cap P$ subtends an angle of at most $\pi/4$ with ν , Lemma 4.4 follows. \square

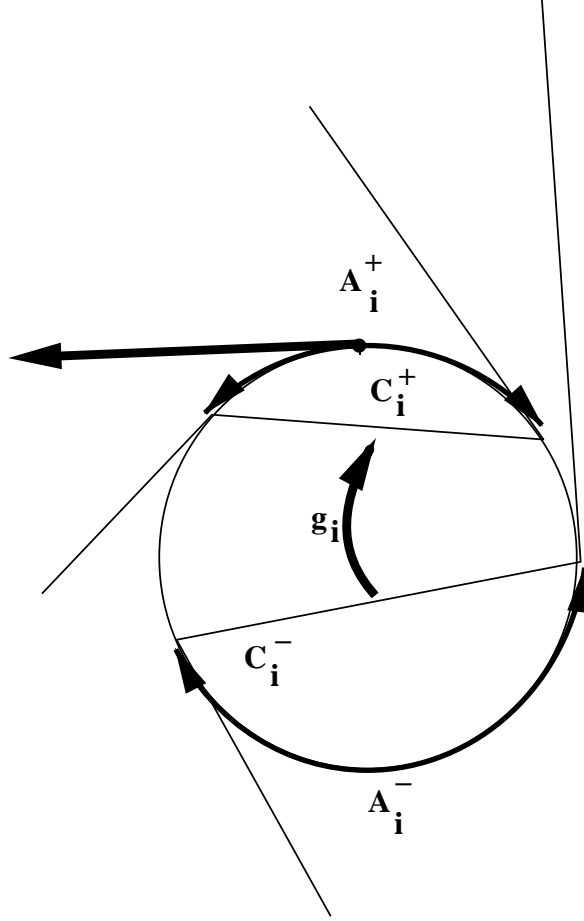


FIGURE 11. Weak-unstable ray in a crooked half-space

Lemma 4.5. *Let $\epsilon > 0$. If γ_k is ϵ -hyperbolic and $\delta < \delta_0$, then*

$$\rho(L_k, L_{k+1}) \geq \frac{\delta\epsilon}{4\sqrt{2}}.$$

PROOF. Apply the Compression Lemma 1.7 with $x \in \gamma_k^{-1}(L_k)$ and $h = \gamma_k$ to obtain

$$B\left(L_k, \frac{\delta\epsilon}{4}\right) \cap E^k(x) \subset \gamma_k B(\gamma_k^{-1}L_k, \delta) = T_k(\delta).$$

Lemma 4.3 implies that the tubular neighborhood $T_k(\delta)$ is disjoint from L_{k+1} . Therefore $B(L_k, \delta\epsilon/4) \cap E^k(x)$ is disjoint from L_{k+1} and

$$(4.5) \quad \rho(x, \partial T_k(\delta)) \leq \rho(x, L_{k+1}) \leq \rho(L_k, L_{k+1}).$$

Lemma 4.4 implies $\angle(\nu, E^k(x) \cap P) < \pi/4$ so

$$\cos \angle(\nu, E^k(x) \cap P) > \frac{1}{\sqrt{2}}.$$

Thus (compare Figure 12)

$$(4.6) \quad \rho(x, \partial T_k(\delta)) = \rho\left(x, \partial B\left(L_k, \frac{\delta\epsilon}{4}\right) \cap E^k(x)\right) \cos \angle(\nu, E^k(x) \cap P) > \frac{\delta\epsilon}{4\sqrt{2}}.$$

Lemma 4.5 follows from (4.5) and (4.6). \square

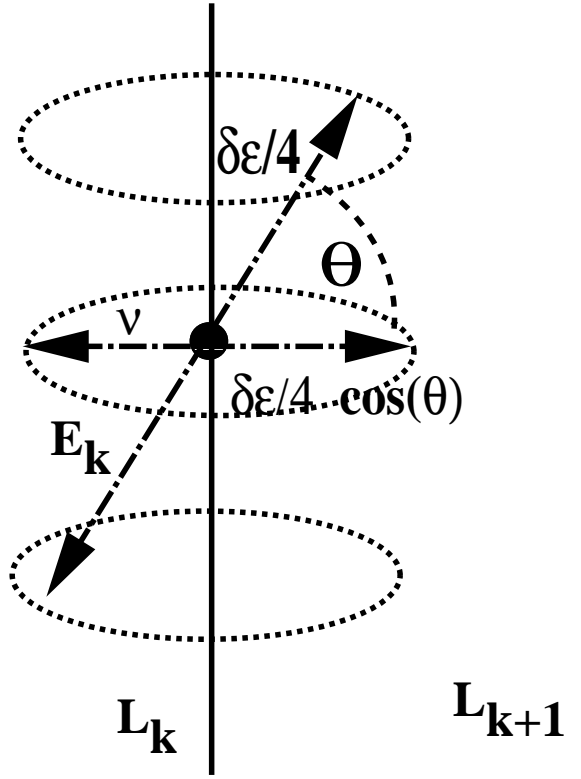


FIGURE 12. Separating the linear approximations

4.5. The alternative to ϵ -hyperbolicity. By Lemma 4.5, it suffices to find $\epsilon > 0$ such that for infinitely many k , the element γ_k is ϵ -hyperbolic. Lemma 2.7 gives a criterion for ϵ -hyperbolicity in terms of the expression of γ_k as a reduced word.

Choose ϵ_0 as in Lemma 2.7 and the sequence $(i_0, j_0), \dots, (i_k, j_k), \dots$ as in Lemma 4.1. Recall that γ_k has the expression

$$\gamma_k = h_{i_0}^{j_0} \dots h_{i_{k-1}}^{j_{k-1}}$$

where $(i_{k+1}, j_{k+1}) \neq (i_k, -j_k)$ for all $k \geq 0$. Lemma 2.7 implies:

Lemma 4.6. *Either g_k is ϵ_0 -hyperbolic for infinitely many k or there exists $k_2 > 0$ such that $(i_k, j_k) = (i_0, -j_0)$ for all $k > k_2$. We may assume that k_2 is minimal, that is, $(i_{k_2}, j_{k_2}) \neq (i_0, -j_0)$.*

Thus g_k has the special form

$$g_k = (g_{i_0}^{j_0})^{k_1} g' (g_{i_0}^{-j_0})^{k-k_1-k_2-1}$$

where $k_1 > 0$ is the smallest k such that $(i_k, j_k) \neq (i_0, j_0)$ and g' is the subword

$$g_{i_{k_1}}^{j_{k_1}} \cdots g_{i_{k_2}}^{j_{k_2}}.$$

g' is the maximal subword of g_k which neither begins with $g_{i_0}^{j_0}$ nor end ends with $g_{i_0}^{-j_0}$. In particular the conjugate of g_k by $\psi = g_{i_0}^{-j_0 k_1}$

$$g_{i_0}^{-j_0 k_1} g_k g_{i_0}^{j_0 k_1} = g' (g_{i_0}^{-j_0})^{k-k_2-1}$$

is ϵ_0 -hyperbolic, by Lemma 2.7.

4.6. Changing the hyperbolicity. The proof concludes by showing that there is a $K > 1$ depending on $g_{i_0}^{-j_0 k_1}$ and taking ϵ smaller than ϵ_0/K , infinitely many g_k are ϵ -hyperbolic for this new choice of ϵ . This contradiction concludes the proof of the theorem.

Lemma 4.7. *Let $\psi \in \mathrm{SO}^0(2, 1)$. Then there exists K such that, for any $\epsilon > 0$, an element $g \in \mathrm{SO}^0(2, 1)$ is ϵ/K -hyperbolic whenever $\psi g \psi^{-1}$ is ϵ -hyperbolic.*

PROOF. Let s denote the distance $d(O, \psi(O))$ that s moves the origin $O \in \mathbb{H}_{\mathbb{R}}^2$ (see (1.2)) and let

$$K = e^s \pi / 2.$$

Since $x^\pm(\psi g \psi^{-1}) = \psi(x^\pm(g))$, it suffices to prove that if $\mathbf{a}_1, \mathbf{a}_2 \in S^1$, then

$$(4.7) \quad K^{-1} \leq \frac{\rho(\psi(\mathbf{a}_1), \psi(\mathbf{a}_2))}{\rho(\mathbf{a}_1, \mathbf{a}_2)} \leq K.$$

Let $\mathrm{SO}(2)$ be the group of rotations and $\mathrm{SO}^0(1, 1)$ the group of *transvections*

$$\tau_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(s) & \sinh(s) \\ 0 & \sinh(s) & \cosh(s) \end{bmatrix}.$$

Since $\mathrm{SO}(2) \subset \mathrm{SO}^0(2, 1)$ is a maximal compact subgroup and $\mathrm{SO}^0(1, 1) \subset \mathrm{SO}^0(2, 1)$ is an \mathbb{R} -split Cartan subgroup, the Cartan decomposition of $\mathrm{SO}^0(2, 1)$ is

$$\mathrm{SO}^0(2, 1) = \mathrm{SO}(2) \cdot \mathrm{SO}^0(1, 1) \cdot \mathrm{SO}(2)$$

and we write $\psi = R_\theta \tau_s R_{\theta'}$ where $s = d(O, \psi(O))$ as above. Since

$$\rho(\psi(\mathbf{a}_1), \psi(\mathbf{a}_2)) = \rho(\tau_s(R_{\theta'}(\mathbf{a}_1)), \tau_s(R_{\theta'}(\mathbf{a}_2)))$$

and

$$\rho(\mathbf{a}_1, \mathbf{a}_2) = \rho(R_{\theta'}(\mathbf{a}_1), R_{\theta'}(\mathbf{a}_2)),$$

it suffices to prove (4.7) for $\psi = \tau_s$.

In this case

$$\frac{d\phi}{\psi^* d\phi} = \frac{1 + \cos \phi}{2} e^s + \frac{1 - \cos \phi}{2} e^{-s}$$

so that

$$e^{-s} \leq \frac{d\phi}{\psi^* d\phi} \leq e^s.$$

Let A be the interval on S^1 joining \mathbf{a}_1 to \mathbf{a}_2 , such that $\Phi(A) \leq \pi$. Its length and the length of its image $\psi(A)$ are given by:

$$\Phi(A) = \int_A |d\phi|, \quad \Phi(\psi(A)) = \int_{\psi(A)} |d\phi| = \int_A \psi^*(|d\phi|).$$

Therefore

$$(4.8) \quad e^{-s} \leq \frac{\Phi(A)}{\Phi(\psi(A))} \leq e^s.$$

Finally the distance $\rho(\mathbf{a}_1, \mathbf{a}_2)$ on S^1 (the length of the chord jointing \mathbf{a}_1 to \mathbf{a}_2) relates to the Riemannian distance by $\rho(\mathbf{a}_1, \mathbf{a}_2) = 2 \sin(\Phi(A)/2)$. Now (since $-\pi \leq \phi \leq \pi$)

$$\frac{2}{\pi} \leq \frac{2 \sin(\phi/2)}{\phi} \leq 1,$$

implies

$$(4.9) \quad \frac{2}{\pi} \leq \frac{\rho(\mathbf{a}_1, \mathbf{a}_2)}{\Phi(A)} \leq 1$$

and

$$(4.10) \quad 1 \leq \frac{\Phi(\psi(A))}{\rho(\psi(\mathbf{a}_1), \psi(\mathbf{a}_2))} \leq \frac{\pi}{2}.$$

Combining inequalities (4.8) with (4.9) and (4.10) implies (4.7). \square

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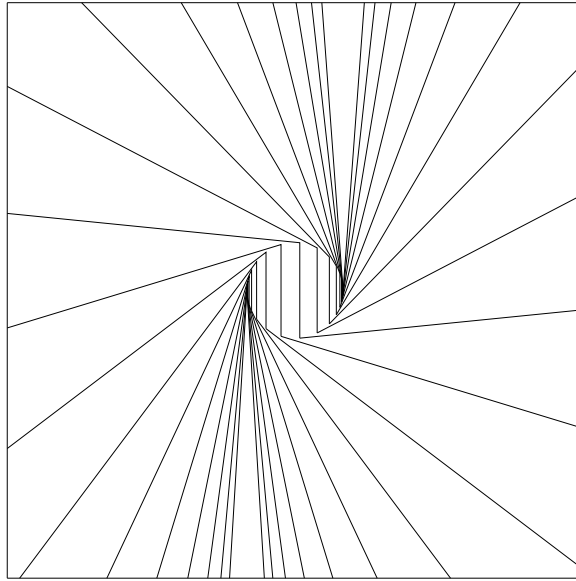


FIGURE 13. Zigzags for a linear hyperbolic cyclic group

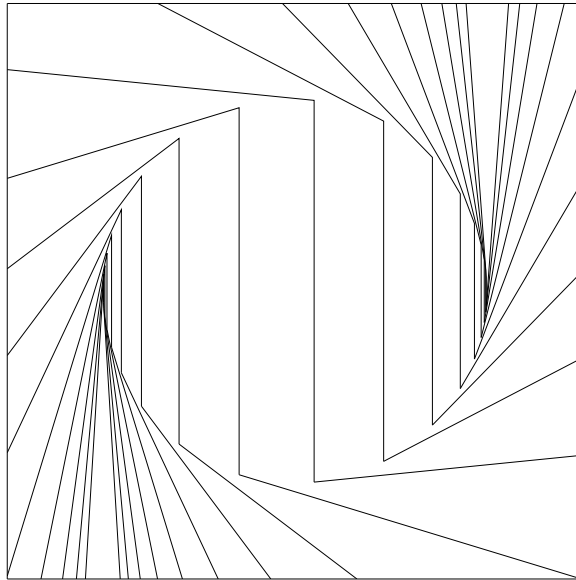


FIGURE 14. Zigzags for a linear hyperbolic cyclic group:close-up

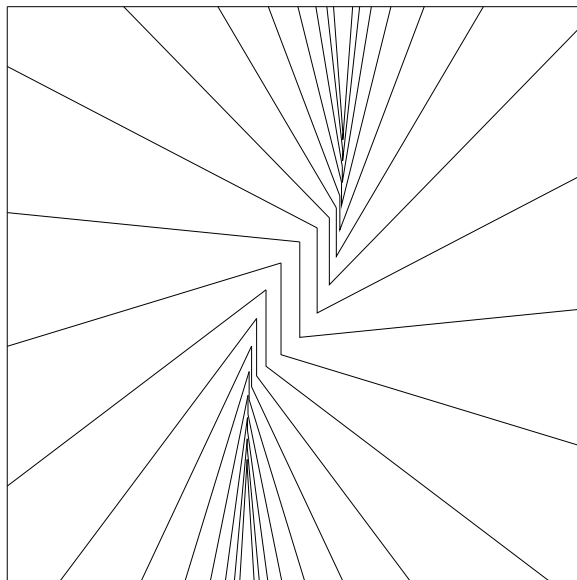


FIGURE 15. Zigzags for an affine hyperbolic cyclic group

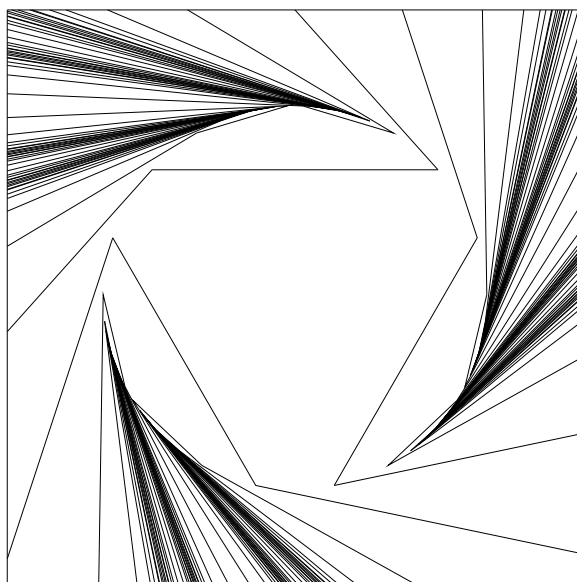


FIGURE 16. Tiling by a discrete linear group

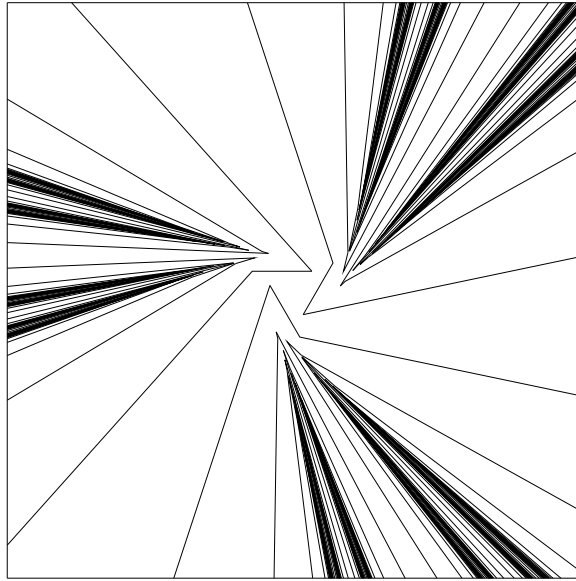


FIGURE 17. Zigzags for an ultra-ideal triangle group: Remote view

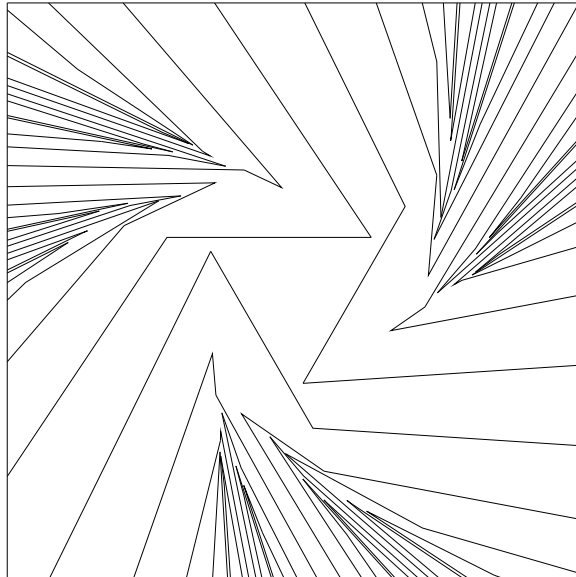


FIGURE 18. Zigzags for an ultra-ideal triangle group: Close-up view